Connecting to Develop Computational Fluency with Fractions

"In fractions, we have one hundred different kinds of rules for solving one hundred different kinds of problems," declared 11-year-old Benny after solving a variety of problems involving fractions (Erlwanger 1973, p. 10). Do your students share Benny's frustration when they add and subtract fractions? Do they think one procedure is required for solving a problem such as 3/8 + 2/8 and a different procedure is required for a problem such as 1/2 + 1/3? Do they view these procedures as being unrelated to each other? Many of my students have held these views. Consequently, they struggled to become computationally fluent with fractions.

NCTM's *Principles and Standards for School Mathematics* (2000) states that all students should develop computational fluency for operating on different types of numbers (whole numbers, fractions, and so on). Such fluency enables students to compute answers to problems in efficient and accurate ways that are meaningful to them. It also enables students to think flexibly about numbers and to make mathematical connections by perceiving similarities or relationships between problems and various solution strategies.

As I prepared to guide a group of intermediate-grade students in exploring the addition and subtraction of fractions, I wondered, "How can I help my students develop this computational fluency described by *Principles and Standards*?" Karen Fuson (2003) suggests that teachers can encourage the development of computational fluency by explicitly focusing students on a "big" or "core" mathematical idea throughout the unit of study. *Principles and Standards* suggests that teachers may also need to continually encourage students to look for mathematical connections between problems and solution strategies. I decided to follow these suggestions and focused my students on the big idea of operating with like-size units (Mack 1995; Smith 2002) while encouraging them to look for similarities as we explored the addition and subtraction of fractions.

By Nancy K. Mack

Nancy Mack, mackn@gvsu.edu, teaches mathematics education courses at Grand Valley State University in Allendale, Michigan. She is interested in the teaching and learning of fractions.

Planning with a Focus on Mathematical Connections

In the past, I guided my students through the sequence of problems traditionally presented in mathematics textbooks. This sequence involves moving back and forth between problems in which fractions have like denominators and those in which fractions have unlike denominators. It also involves gradually increasing the size of the numbers in problems from less than one to greater than one. Although this sequence is based on a sound theoretical mathematical perspective, I often found the rapid movement between problems with like and unlike denominators to be too big of a cognitive jump for many of my students. These students, some of whom had special learning needs in mathematics, had not readily transferred the idea of operating on like-size units from problems with like denominators to those with unlike denominators. I thought my current group of students was likely to have the same difficulty.

In planning, I focused on the following questions:

- What types of problems would help my students

readily focus on the idea of operating on like-size units?
- What types of problems would help them gradually build on and extend this big idea in small steps?
- What makes problems mathematically connected or similar to one another from my students' perspective?

I decided to let my students' thinking guide the sequencing of addition and subtraction problems instead of strictly adhering to the sequence that our textbook presented. The work of mathematics educators who have looked closely at children's early learning of fraction concepts and operations, such as Empson (1995) and Mack (1990, 1993), guided my interpretation of my students' responses. These works also provided guidance for possible ways to sequence addition and subtraction problems based on my students' thinking.

My students progressed through the addition and subtraction of fractions by first exploring problems with like denominators. These problems helped make the idea of operating on like-size units readily apparent to the children. Following this, my students explored problems that gradually increased in complexity as they worked their way toward problems involving unlike denominators. **Figure 1** shows the specific sequence that resulted.

Another important part of my planning focused on the instructional approach, which reflected NCTM's *Principles and Standards*. We made extensive use of word problems and manipulative materials in the form of fraction circles and strips. We wrote number sentences (for example, 3/8 + 2/8) as mathematical representations for word problems and I encouraged the students to solve the problems in ways that were meaningful to them. Additionally, I guided my students to identify and represent fractions in a way that helped them con-

Figure 1

Sequence progression of addition and subtraction problems

Generalized Mathematical Form of Problem	Problem Example
$a/b \pm c/b = d/b$, where $d < b$	3/8 + 2/8 =5/8, 3/8 – 2/8 =1/8
$a/b + c/b = n\ d/b$, where $n = 1$	5/8 + 7/8 = 1 4/8
$1 - c/b = d/b$	1 – 3/8 = 5/8
$n - c/b = m\ d/b$, where $n > 1$ and $m \geq 1$	4 – 3/8 = 3 5/8
$n\ 1/b - c/b = m\ d/b$, where $n > 1$, $m \geq 1$ and $c > 1$	4 1/8 – 3/8 = 3 6/8
$n\ 1/b - y\ c/b = m\ d/b$, where n & $y > 1$, $m \geq 1$ and $c > 1$	4 1/8 – 2 3/8 = 1 6/8
$n\ a/b - y\ c/b = m\ d/b$, where n & $y > 1$, $m \geq 1$ and $a < c$	4 3/8 – 2 5/8 = 1 6/8
$1/b \pm 1/g = a/g$, where g is a multiple of b	1/4 + 1/8 = 3/8, 1/4 – 1/8 = 1/8
$1/b \pm 1/g = a/z$, where b and g are relatively prime	1/2 + 1/3 = 5/6, 1/2 – 1/3 = 1/6
$a/b \pm c/g = x/z$, where a & $c > 1$ and b and g are relatively prime	3/4 + 2/3 = 1 5/12, 3/4 – 2/3 = 1/12
$n\ a/b \pm y\ c/g = m\ x/z$, where n & $y \geq 1$, a & $c \geq 1$, and b and g are relatively prime	4 1/3 + 1 1/2 = 5 5/6 4 1/3 – 1 1/2 = 2 5/6

sider not only the number of pieces in a whole but also the fractional size of the pieces involved in a problem; for example, 3/4 can be thought of as three one-fourths. I did this by asking, “What size is each of these individual pieces?” as we used manipulative materials to represent fractions prior to our work with addition and subtraction.

Making Connections When Working with Like Denominators

My students readily focused on the idea of operating on like-size units as they solved problems involving adding or subtracting fractions with like denominators and subtracting a fraction from a whole number. This focus enabled them to determine appropriate ways to solve problems that were resistant to common misconceptions, such as adding denominators together when adding fractions. This focus also helped the students see similarities between solution strategies involving the same or different forms of representation and similarities between increasingly complex problems. As the students perceived these similarities, they examined the efficiency of various strategies through their own efforts.

The following dialogue illustrates how readily my students focused on the idea of like-size units and saw similarities between problems and solution strategies with limited assistance from me.

Me. I have a problem for you to solve. You have three-eighths of a medium pepperoni pizza. I give you two-eighths more of a medium pepperoni pizza. How much of a medium pepperoni pizza do you have now?

Greg. I have five-eighths of a pepperoni pizza.

Me. How do you know?

Greg. See, here’s a circle made of eighths *(makes a whole circle with fraction pieces; see* ***fig. 2****)*. This is three one-eighths *(moves three-eighths away from the whole circle)* and this is two one-eighths *(moves two-eighths away)*, and if I put them together there’s five one-eighths. One, two, three, four, five *(counting)*. They’re all eighths so it’s five-eighths.

(Other students share their strategies. All agree that the answer is five-eighths.)

Me. What if I said I think the answer is five-sixteenths? *(Writes “3/8 + 2/8 = 5/16.” A stunned silence occurs in the classroom, followed by questions of “Huh?” Students appear to be deep in thought during a long pause.)*

Me. Three plus two is five, right?

Students. Right.

Me. And eight plus eight is sixteen, right? So it's five-sixteenths.

Brian. I don't think that's right. It can't be five-sixteenths.

Me. Why can't it?

Brian. Because these are eighths *(holds up one-eighth of a fraction circle)*. If you put them together you still have eighths *(shows this in a manner similar to Greg's)*. See, you didn't make them into sixteenths when you put them together. They're still eighths.

Me. Let's try this problem. You have a board that is three yards long. You cut off a piece that is five-sixths of a yard long to make a shelf for your room. How much of the board do you have left?

Samantha. Oh, this is tricky!

Me. It is? Why is it tricky?

Samantha. Well, because it just is. Because we haven't done these kinds before, ones where they're bigger than one.

Me. I think you can do it. Think about what you know and problems you already know how to solve.

(Students work independently or with a partner on the problem.)

Nathan. I think I have two and one-sixth yards left. I did it like this. First, I wrote three minus five-sixths equals what on my paper. Then I thought, "There's six-sixths in one whole; there's twelve in two, and eighteen in three. Eighteen-sixths." I wrote eighteen-sixths minus five-sixths because I know how to solve it when it's like this [same denominators]. Eighteen minus five is thirteen. So I had thirteen-sixths. I know that's more than one whole yard because six-sixths is one, so seven-sixths would be left. That's another one because it's six-sixths and there's still one-sixth left, so it's two and one-sixth *(see **fig. 3a**)*.

Alissa. I got two and one-sixth but I did it a different way. I took five-sixths from one yard and had one-sixth left of that yard because one yard is six-sixths. I still had two yards left because I started with three yards, so I have two and one-sixth yards left. *(See **fig. 3b** for a mathematical illustration of Alissa's thinking.)*

Me. What do the rest of you think about Nathan's and Alissa's strategies?

Madison. I think both ways are right. They both made the pieces the same size. Nathan cut them all up into sixths first and Alissa only cut up one, but Nathan only used part of one and then put the pieces back together to make wholes. I did it Nathan's way but I made a mistake. I added wrong but my way still works. I like Alissa's way better.

Me. Why do you like Alissa's strategy better?

Madison. It's faster and it's easier. You only have to cut up one whole to get the pieces you need so you might not make as many mistakes.

As the dialogue suggests, my students' focus on the idea of operating on like-size units played a valuable role in their ability to solve problems and see similarities between solution strategies. It also helped them examine the efficiency of their strategies. Consequently, my students quickly abandoned less efficient strategies in favor of those they deemed more efficient as they made mathematical connections.

Making Connections When Working with Unlike Denominators

My students continued to focus on the idea of like-size units and to look for similarities when they

Figure 2

Greg's solution to problem corresponding to "3/8 + 2/8 = ?"

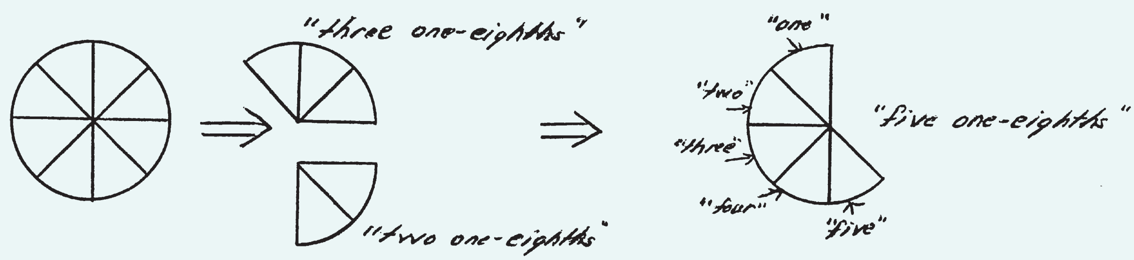
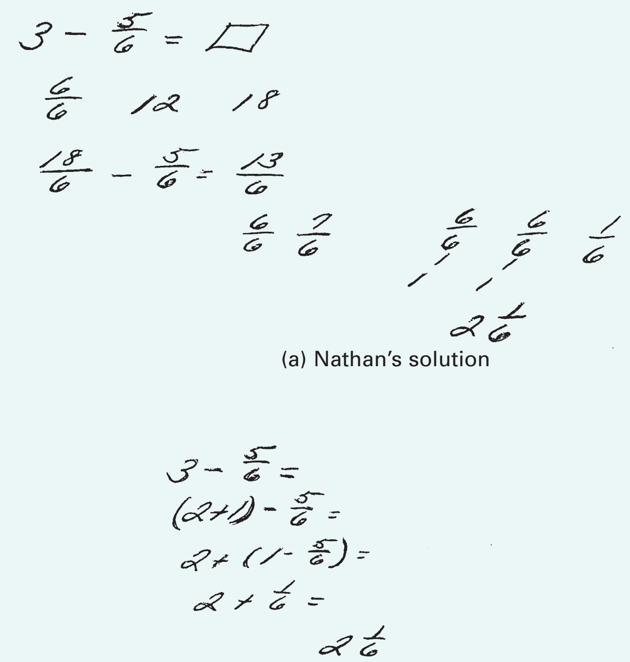


Figure 3

Solutions to problem corresponding to "3 – 5/6 = ?"



(a) Nathan's solution

(b) Alissa's solution

encountered problems involving fractions with unlike denominators. This focus enabled them to suggest that they needed like-size units to solve the problems; however, what size units were needed in each problem was not immediately obvious to any of my students. How they could obtain these units was also not obvious. We had not yet fully explored the concept of equivalent fractions. We had addressed only fractions equivalent to one and to one-half as they emerged in problems.

I needed to help my students realize that fractions can be renamed in terms of other fractions. I also needed to guide them to realize that renaming fractions when solving a problem can be helpful. I helped my students comprehend these ideas by making explicit references to strategies they previously used to solve problems. I also encouraged them to look for similarities between problems and similarities between strategies involving the use of manipulative materials and those involving number sentences.

As I guided my students in this way, they found appropriate like-size units and solved problems by using strategies that were based on their prior knowledge of fractions. They also drew on their knowledge of operating on like-size units to determine similarities between solution strategies. The following dialogue illustrates these ideas.

Me. Try this problem. Yesterday afternoon you started reading a library book. You read one-half of the library book yesterday. You read one-third of the book earlier today. How much of the library book have you now read?

(Long pause. Students have puzzled looks on their faces.)

Me. Why aren't you immediately telling me the answer to this problem? You did that with lots of the other problems we solved.

Samantha. This one's different.

Me. How is it different? What's different about it?

Mike. The numbers are different. The pieces are different *(holds up 1/2 and 1/3 with the fraction strips)*. They're not the same size like they were before.

Me. But you solved problems before when the pieces weren't the same size. Remember the problem about the board and making a shelf for your room? Were the pieces the same size? How did you solve that problem?

Alexis. I changed the three. I made it two and six-sixths so there'd be pieces that were the same size.

Me. So you renamed a number . . .

Donovan (interrupting). Can you rename fractions?

Me. What do you think?

Brian. See, one-half and two-fourths are the same *(holds up one-half and two-fourths with fraction circles and places the fourths on top of the half)*. Is that what you mean when you say rename?

Me. Yes, that's exactly what I mean by renaming. Do you think renaming something might help you solve the problem like it did before?

(Students explore with the fraction circles and strips.)

Nathan. I don't see how to do this. I can't make thirds into halves and I can't make halves into thirds, at least not evenly.

Alexis. Can we rename both fractions?

Me. Why do you ask that? Why do you want to rename both fractions?

Alexis. Well, before we only did one, but look, see, I got one-half and three-sixths are the same as that, and one-third and two-sixths are the same. So if they can all be sixths, I can solve the problem because I know how to solve it when the pieces are the same size.

Me. Do the rest of you think it's okay to rename one or more fractions to help you solve the problem?

Madison. Well, if we don't we still got different-size pieces and we probably can't tell how much we have.

Me. Let's try this one. Let's try a problem where I give you only the number sentence and not a word problem. Try this *(writes "3/4 + 7/8 = ?" on board).*

Alissa. It's one and five-eighths. I thought of the circles in my head. Fourths are bigger than eighths, but if I cut each of the fourths in two, I'd have eighths and there'd be six pieces of eighths, six one-eighths. See, I thought three-fourths is the same as six-eighths. Then I took one-eighth from the six-eighths and put it with the seven-eighths to make one whole, and there was five-eighths left. *(See* ***fig. 4a*** *for a mathematical illustration of Alissa's thinking.)*

Mike. I got the same answer but I did it a different way. I used the circles and then wrote down what I did with the circles. I took two-eighths from the seven-eighths and put it with three-fourths so I'd have a whole, because two-eighths and one-fourth are the same so if I already had three and I got another one, I'd have four-fourths, one whole, and there's five-eighths left. *(See* ***fig. 4b*** *for Mike's solution.)*

Me. How are these two strategies the same?

Greg. Well, first they made the fractions into pieces that were the same size. That's what you have to do if you're gonna solve problems and they've got fractions. You turn it into a problem you already know how to solve.

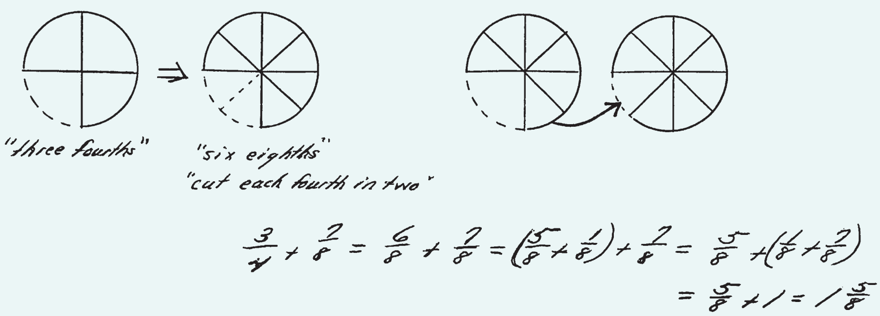
As the dialogue illustrates, my students' primary struggles with problems involving unlike denominators centered on the idea of renaming fractions. After they realized that fractions could be renamed, they drew on the idea of operating on like-size units to solve problems and viewed strategies for different problems as being related to one another.

Summary

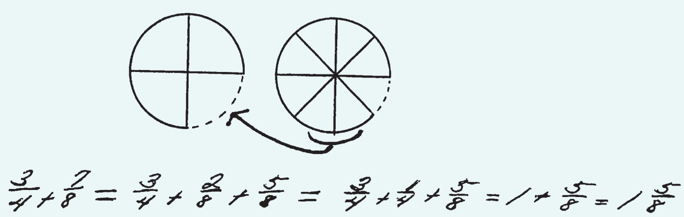
Encourage your students to focus on the fractional size of pieces involved as they explore fractions; for example, 3/8 can be thought of as three one-eighths. Thinking of fractions in this way may help them realize that they are operating on like-size units as they add and subtract fractions. It may also help them make mathematical connections by perceiving similarities between different types of problems and various solution strategies.

Figure 4

Solutions to "3/4 + 7/8 = ?"



(a) Alissa's solution



(b) Mike's solution

Listen carefully to your students to determine whether the sequence in which they encounter problems involving addition and subtraction of fractions aides them in making connections and developing computational fluency. Do their responses suggest they see similarities between different types of problems and various solution strategies? If not, do their responses suggest that there is too big of a cognitive jump for them between different types of problems? If this is the case, try to examine the sequencing of problems from your students' perspective. What mathematical ideas do they focus on when they add and subtract fractions? What do the students focus on when they view problems as being related to one another? Do your students need to explore one idea in depth with simple problems before extending it to more complex problems? If this is the case, try presenting your students with a string of closely related problems that gradually extends a big mathematical idea in very small steps. This string should be based on the students' perspective and may consequently differ from the sequence of problems that textbooks traditionally present. This may be particularly important if your students are greatly challenged by mathematics or have special learning needs in mathematics.

Encourage your students to make mathematical connections by frequently asking them to look for similarities between different types of problems and between solution strategies. You might find it helpful to ask questions such as the following:

- How is this problem similar to ones you previously solved?
- How is this problem different?
- What is the same about these solution strategies?
- Do you think you could use what you did in a problem you solved before to help you solve this problem?

By encouraging our students to focus on a big mathematical idea and to look for connections between problems and solution strategies, we may help them view the addition and subtraction of fractions in a unified manner. This unified view suggests that our students are developing computational fluency with fractions.

Correction

In the August 2004 issue of *TCM*, an incorrect unit of measurement was used in the "Then and Now" problem that appeared on page 25 of the "Math by the Month" department. We apologize for the error. Thank you to Mary Lou Ferro for drawing it to our attention. The corrected problem reads as follows:

WEEKLY ACTIVITIES

LEWIS AND CLARK: 5–6 **AUGUST 2004**

Then and now. The Louisiana Purchase was made for $15,000,000. The amount of land purchased was about 375 million acres. At that price, about how much money per acre did the United States spend to purchase the land from Napoleon? If the average price of land today were $3500 per acre, how much would the United States need to pay for the Louisiana Purchase?

References

Empson, Susan B. "Using Sharing Situations to Help Children Learn Fractions." *Teaching Children Mathematics* 2 (October 1995): 110–14.

Erlwanger, Stanley H. "Benny's Conceptions of Rules and Answers in IPI Mathematics." *Journal of Children's Mathematical Behavior* (1973): 7–25.

Fuson, Karen C. "Developing Mathematical Power in Whole Number Operations." In *A Research Companion to Principles and Standards for School Mathematics*, edited by Jeremy Kilpatrick, W. Gary Martin, and Deborah Schifter, pp. 68–94. Reston, Va.: National Council of Teachers of Mathematics, 2003.

Mack, Nancy K. "Learning Fractions with Understanding: Building on Informal Knowledge." *Journal for Research in Mathematics Education* 21 (January 1990): 16–32.

———. "Making Connections to Understand Fractions." *Arithmetic Teacher* 40 (February 1993): 362–64.

———. "Confounding Whole-Number and Fraction Concepts When Building on Informal Knowledge." *Journal for Research in Mathematics Education* 26 (November 1995): 422–41.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, Va.: NCTM, 2000.

Smith, John P. III. "The Development of Students' Knowledge of Fractions and Ratios." In *Making Sense of Fractions, Ratios, and Proportions*, 2002 Yearbook of the National Council of Teachers of Mathematics, edited by Bonnie Litwiller and George Bright, pp. 3–17. Reston, Va.: National Council of Teachers of Mathematics, 2002. ▲